MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

American Mathematics Competitions

26th Annual

AMC 8

American Mathematics Contest 8

Tuesday, November 16, 2010

This Solutions Pamphlet gives at least one solution for each problem on this year's exam and shows that all the problems can be solved using material normally associated with the mathematics curriculum for students in eighth grade or below. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers will share these solutions with their students. However, the publication, reproduction, or communication of the problems or solutions of the AMC 8 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. *Dissemination at any time via copier, telephone, email, internet or media of any type is a violation of the competition rules.*

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1. **Answer (C):** The total number of students taking the test is $11+8+9=28$.

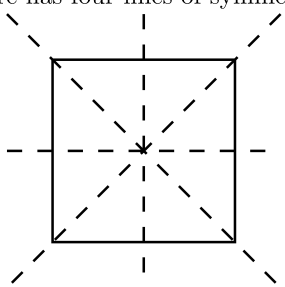
2. **Answer (D):** $5*10=\frac{5\times10}{5+10}=\frac{50}{15}=\frac{5\times10}{5\times3}=\frac{10}{3}$.

3. **Answer (C):** The highest price in January was \$17 and the lowest in March was \$10. The \$17 price was \$7 more than the \$10 price, and 7 is 70% of 10.

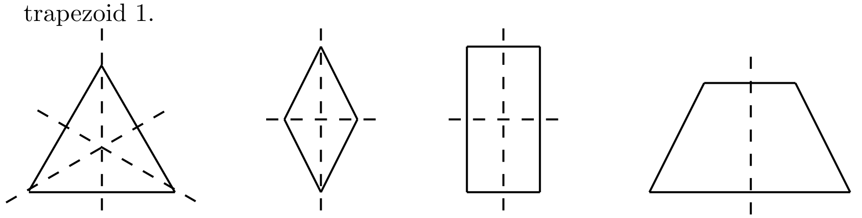
4. **Answer (C):** Arrange the numbers in increasing order: $0,0,1,2,3,3,3,4$. The mean is the sum divided by 8, or $\frac{16}{8}=2$. The median is halfway between 2 and 3, or 2.5. The mode is 3, because there are more 3's than any other number. The sum of the mean, median, and mode is $2+2.5+3=7.5$.

5. **Answer (B):** The ceiling is $2.4-1.5=0.9$ meters $=90$ centimeters above Alice's head. She can reach 46 centimeters above the top of her head, and the light bulb is 10 centimeters below the ceiling, so the stool is $90-46-10=34$ centimeters high.

6. **Answer (E):** A square has four lines of symmetry.



The number of lines of symmetry for the other figures are:
equilateral triangle 3, non-square rhombus 2, non-square rectangle 2, isosceles trapezoid 1.



7. **Answer (B):** Four pennies are needed to make small change. Adding two nickels means that change up to fourteen cents can be made as efficiently as possible. Adding a dime extends the efficiency up to 24 cents. Adding three quarters permits any amount of change up to \$0.99 .

$$4 \text{ pennies} + 2 \text{ nickels} + 1 \text{ dime} + 3 \text{ quarters} = 10 \text{ coins worth } \$0.99$$

Note that 4 pennies, 1 nickel, 2 dimes, and 3 quarters also satisfies the requirements. For 10 to be the smallest number of coins, one must prove that 9 coins will not work. Freddie will still need at least 4 pennies and 1 nickel to make any amount up to 9. Freddie will still need at least 1 more nickel and 1 dime to make any amount up to 24. If Freddie chose only 2 quarters, then he would need at least 9 other coins (4 dimes, 1 nickel, and 4 pennies) to reach 99 cents. He would need at least 11 coins. Thus 10 is the smallest number of coins as shown above.

8. **Answer (D):** Emily gains on Emerson at the rate of $12 - 8 = 4$ miles per hour. To get from $\frac{1}{2}$ mile behind Emerson to $\frac{1}{2}$ mile in front of him, she must gain 1 mile on him. This takes $\frac{1}{4}$ hour, which is 15 minutes.

OR

Make a chart that shows the position of Emily and Emerson every 5 minutes. Note that for every 5 minutes Emily rides $\frac{12}{60}$ miles per minute $\times$ 5 minutes $= 1$ mile, and Emerson skates $\frac{8}{12}$ mile. Eventually, Emily will be $\frac{1}{2}$ mile ahead of Emerson. Notice that happens after 15 minutes.

Time (minutes)	0	5	10	15
Emily's Distance (miles)	0	1	2	3
Emerson's Distance (miles)	$\frac{1}{2}$	$1\frac{1}{6}$	$1\frac{5}{6}$	$2\frac{1}{2}$

9. **Answer (D):** The three tests contain a total of 75 problems. Ryan received 80% of $25 = 20$, 90% of $40 = 36$, and 70% of $10 = 7$. Ryan correctly answered $20 + 36 + 7$ problems, for a total of 63 problems. The percent of problems Ryan answered correctly was:

$$\frac{63}{75} = \frac{21}{25} = \frac{21 \cdot 4}{25 \cdot 4} = \frac{84}{100} = 84\%$$

10. **Answer (B):** If six pepperonis fit across the diameter, then each pepperoni circle has a diameter of 2 inches and a radius of 1 inch. The area of each

pepperoni is $\pi(1)^2 = \pi$ square inches. The 24 pepperoni circles cover 24π square inches of the pizza. The area of the pizza is $\pi(6)^2 = 36\pi$ square inches. The fraction of the pizza covered by pepperoni is $\frac{24\pi}{36\pi} = \frac{2}{3}$.

11. **Answer (B):** The sum of the heights of the two trees can be divided into 7 parts where one part is 16 feet. The taller tree has 4 parts so its height is $4 \times 16 = 64$ feet.

12. **Answer (D):** The number of blue balls in the bag is 20% of 500, which is $(0.20)(500) = 100$. After some red balls are removed, the 100 blue balls must be 25%, or $\frac{1}{4}$, of the number in the bag. There must be $(4)(100) = 400$ balls in the bag, so $500 - 400 = 100$ red balls must be removed.

OR

Let x represent the number of red balls removed. Remember that $0.80(500) = 400$ is the number of red balls that were originally in the bag. Setup and solve the proportion:

$$\begin{aligned} 75\% = \frac{3}{4} &= \frac{400 - x}{500 - x} \\ 3(500 - x) &= 4(400 - x) \\ 1500 - 3x &= 1600 - 4x \end{aligned}$$

The number of red balls removed is $x = 100$.

13. **Answer (E):** One strategy is to try the choices:

$$\begin{array}{ll} 5 + 6 + 7 = 18; & 5 \neq 30\% \text{ of } 18 \\ 6 + 7 + 8 = 21; & 6 \neq 30\% \text{ of } 21 \\ 7 + 8 + 9 = 24; & 7 \neq 30\% \text{ of } 24 \\ 8 + 9 + 10 = 27; & 8 \neq 30\% \text{ of } 27 \\ 9 + 10 + 11 = 30; & 9 = 30\% \text{ of } 30 \end{array}$$

If the shortest side is 9, then the longest side is 11.

OR

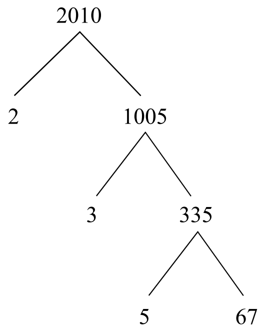
Let the three consecutive integers be side lengths $x, x - 1$, and $x - 2$.

$$\begin{aligned} x - 2 &= 0.3(x + x - 1 + x - 2) \\ x - 2 &= 0.3(3x - 3) \\ x - 2 &= 0.9x - 0.9 \\ 0.1x &= 1.1 \end{aligned}$$

$$x = 11$$

The longest side is 11.

14. **Answer (C):** The prime factors of 2010 are: 2, 3, 5, 67



The sum of the prime factors is $2 + 3 + 5 + 67 = 77$.

15. **Answer (C):** The 30 green gum drops are $100\% - (30+20+15+10)\% = 25\%$ of the total gum drops, so there are 120 gum drops in the jar. The number of blue gum drops is 30% of 120, which is 36, and the number of brown gum drops is 20% of 120, which is 24. After half the blue gum drops are replaced by brown ones, the number of brown gum drops is $24 + \frac{1}{2}(36) = 42$.

16. **Answer (B):** Let the radius of the circle be 1. Then the area of the circle is $\pi(1)^2 = \pi$. The area of the square is π, so its side length $\sqrt{\pi}$. The ratio of the side length of the square to the radius of the circle is $\frac{\sqrt{\pi}}{1} = \sqrt{\pi}$.

    Note: The "squaring of a circle" is a classical problem. In the latter part of the 19th century it was proven that a square having an area equal to that of a given circle cannot be constructed with the standard tools of straightedge and compasss because it is impossible to construct a transcendental number, e. g. $\sqrt{\pi}$.

17. **Answer (D):** The area below $\overline{PQ}$ is

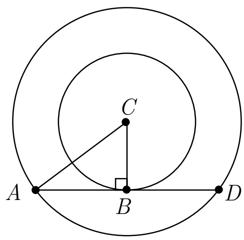
$$\begin{aligned} 1 + \frac{1}{2} \cdot 5 \cdot (1 + QY) &= 5 \\ \frac{5}{2} \cdot (1 + QY) &= 4 \\ 1 + QY &= \frac{8}{5} \end{aligned}$$

$$QY = \frac{3}{5}$$

Then $XQ = 1 - QY = 1 - \frac{3}{5} = \frac{2}{5}$, so $\frac{XQ}{QY} = \frac{2/5}{3/5} = \frac{2}{3}$.

18. **Answer (C):** The given ratio implies that $AD = \dfrac{3}{2}AB = \dfrac{3}{2} \cdot 30 = 45$ inches, so the area of the rectangle is $AB \cdot AD = 30 \cdot 45$ square inches. The 2 semicircles make 1 circle with radius $= 15$ inches. The area of the circle is $15^2\pi$ square inches. The ratio of the areas is $\dfrac{30 \cdot 45}{15 \cdot 15\pi} = \dfrac{6}{\pi}$.

19. **Answer (C):** In the right triangle ABC, AB is 8. By the Pythagorean Theorem, $8^2 + BC^2 = 10^2$ so $BC = 6$. The area of the outer circle is $10^2\pi = 100\pi$ and the area of the inner circle is $6^2\pi = 36\pi$. The area between the circles is $100\pi - 36\pi = 64\pi$.



20. **Answer (A):** Because $\frac{2}{5}$ and $\frac{3}{4}$ of the people in the room are whole numbers, the number of people in the room is a multiple of both 5 and 4. The least common multiple of 4 and 5 is 20, so the minimum number of people in the room is 20. If $\frac{2}{5}$ of 20 people are wearing gloves, then 8 people are wearing gloves. If $\frac{3}{4}$ of 20 people are wearing hats, then 15 are wearing hats. The minimum number wearing gloves and hats occurs if the 5 not wearing hats are each wearing gloves. This leaves $8 - 5 = 3$ people wearing both gloves and hats.

OR

If 8 are wearing gloves and 15 are wearing hats, then $8 + 15$ are wearing gloves and/or hats. There is a minimum of 20 people in the room, so $23 - 20 = 3$ people are wearing both a hat and gloves.

21. **Answer (C):** Reason backward as follows. On the third day, Hui reads $\frac{1}{3}$ of the remaining pages plus 18 more, leaving her with 62 pages left to read. This means that $62 + 18 = 80$ is $\frac{2}{3}$ of the number of pages remaining at the end of the

second day. She had $\frac{3}{2} \times 80 = 120$ pages left to read at the end of the second day. On the second day she read $\frac{1}{4}$ of the remaining pages plus 15 more, so $120 + 15 = 135$ is $\frac{3}{4}$ of the number of pages remaining at the end of the first day. She had $\frac{4}{3} \times 135 = 180$ pages left to read at the end of the first day. On the first day she read $\frac{1}{5}$ of the pages plus 12 more. So $180 + 12 = 192$ is $\frac{4}{5}$ of the number of pages in the book. The total number of pages is $\frac{5}{4} \times 192 = 240$.

OR

Setup a chart working backwards through the days, to show the number of pages that Hui has left to read each day.

Day	Extra Pages Read	Fraction Read	Fraction Left	Calculation	Pages to be Read
4	0				62
3	18	1/3	2/3	$(62 + 18)/(2/3)$	120
2	15	1/4	3/4	$(120 + 15)/(3/4)$	180
1	12	1/5	4/5	$(180 + 12)/(4/5)$	240

22. **Answer (E):** Take 452 as the original number, and subtract 254. The difference is 198 and the units digit is 8. The same result will be obtained with any number that meets the criteria.

OR

The original number is $100a + 10b + c$, with $a - c = 2$. The reversed number will be $100c + 10b + a$. The difference is $99a - 99c = 99(a - c) = 99(2) = 198$. So the units digit is 8.

23. **Answer (B):** By the Pythagorean Theorem, the radius OQ of circle O is $\sqrt{2}$. Given the coordinates P, Q, R, and S, the diameters $\overline{PQ}$ and $\overline{RS}$ of the semicircles have length 2. So the areas of the two semicircles will equal the area of a circle of radius 1. Thus the desired ratio is $\dfrac{\pi \cdot 1^2}{\pi(\sqrt{2})^2} = \dfrac{1}{2}$.

24. **Answer (A):** $2^{24} = (2^8) \cdot (2^{16}) = (2^8) \cdot (4^8) < (2^8) \cdot (5^8) = 10^8 = (4^4) \cdot (5^8) < (5^4) \cdot (5^8) = 5^{12}$.

25. **Answer (E):** Jo can climb 1 stair in 1 way, 2 stairs in 2 ways: $1 + 1$ or 2, and 3 stairs in 4 ways: $1 + 1 + 1$, $1 + 2$, $2 + 1$ or 3. Jo can start a flight of 4 stairs with 1 stair, leaving 3 stairs to go and 4 ways to climb them, or with 2 stairs, leaving 2 stairs to go and 2 ways to climb them or with 3 stairs, leaving 1 to

go and 1 way to climb it. This means there are $1 + 2 + 4 = 7$ ways to climb 4 stairs. By the same argument, there must be $2 + 4 + 7 = 13$ ways to climb 5 stairs and $4 + 7 + 13 = 24$ ways to climb 6 stairs.

OR

Making a systematic list is another approach.

Number of stair moves	Stair move sequence	Count
6	1-1-1-1-1-1	1
5	1-1-1-1-2; 1-1-1-2-1; 1-1-2-1-1; 1-2-1-1-1; and 2-1-1-1-1	5
4	1-1-1-3; 1-1-3-1; 1-3-1-1; 3-1-1-1; 1-1-2-2; 1-2-1-2; 1-2-2-1; 2-1-1-2; 2-1-2-1; and 2-2-1-1	10
3	1-2-3; 1-3-2; 2-1-3; 2-3-1; 3-1-2; 3-2-1; and 2-2-2	7
2	3-3	1

The number of ways Jo can climb the stairs is $1 + 5 + 10 + 7 + 1 = 24$.

The problems and solutions for this contest were proposed by Steve Blasberg, Thomas Butts, John Cocharo, Steve Davis, Melissa Desjarlais, Steven Dunbar, Margie Raub Hunt, Joe Kennedy, Norbert Kuenzi, Sister Josanne Furey, Jeganathan Sriskandarajah, David Wells, LeRoy Wenstrom and Ron Yannone.

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